

Nalezněte diferenciál funkce

$$df(x, y) = f_x(x, y) dx + f_y(x, y) dy, \quad (dx = x - x_0 = h, dy = y - y_0 = k)$$

1. $f(x, y) = x^3y - 2xy^2 + 5x$
 $f_x = 3x^2y - 2y^2 + 5, \quad f_y = x^3 - 4xy$
 $df(x, y) = (3x^2y - 2y^2 + 5)dx + (x^3 - 4xy)dy$
2. $f(x, y) = e^{x^2-y}, \quad a = (1, 0)$
 $f_x = e^{x^2-y} 2x, \quad f_y = e^{x^2-y}(-1)$
 $df(x, y) = (e^{x^2-y} 2x)dx - (e^{x^2-y})dy, \quad df(1, 0) = 2e dx - e dy$
3. $f(x, y) = \arcsin \frac{x}{\sqrt{x^2+y^2}}, \quad a = (\sqrt{3}, 1)$
 $f_x = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\sqrt{x^2+y^2}-x \cdot \frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\sqrt{x^2+y^2}}{y} \cdot \frac{y^2}{x^2+y^2} = \frac{y}{x^2+y^2}$
 $f_y = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \left(-\frac{1}{2}\right) \frac{x}{\sqrt{x^2+y^2}^3} 2y = -\frac{\sqrt{x^2+y^2}}{y} \frac{xy}{\sqrt{x^2+y^2}^3} = \frac{-x}{x^2+y^2}$
 $df(x, y) = \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy, \quad df(\sqrt{3}, 1) = \frac{1}{4} dx - \frac{\sqrt{3}}{4} dy$

Pomocí diferenciálu přibližně vypočtete

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

1. $\sqrt{0,98^2 + 2,05^3}$
 $x_0 = 1, y_0 = 2, dx = -0.02, dy = 0.05$
 $f(x, y) = \sqrt{x^2 + y^3}, f(1, 2) = 3$
 $f_x = \frac{2x}{2\sqrt{x^2+y^3}} = \frac{x}{\sqrt{x^2+y^3}}, \quad f_y = \frac{3y^2}{2\sqrt{x^2+y^3}}$
 $df(x, y) = \frac{x}{\sqrt{x^2+y^3}} dx + \frac{3y^2}{2\sqrt{x^2+y^3}} dy, \quad df(1, 2) = \frac{1}{3} dx + 2 dy$
 $\sqrt{0.98^2 + 2.05^3} \approx 3 + \frac{1}{3}(-0.02) + 2(0.05) = 3 - \frac{2}{300} + \frac{10}{100} = \frac{900-2+30}{300} = \frac{928}{300}$
2. $e^{0.04^3-0.01}$
 $x_0 = 0, y_0 = 0, dx = 0.04, dy = 0.01$
 $f(x, y) = e^{x^3-y}, f(0, 0) = 1$
 $f_x = e^{x^3-y} 3x^2, \quad f_y = e^{x^3-y}(-1)$
 $df(x, y) = 3x^2 e^{x^3-y} dx - e^{x^3-y} dy, \quad df(0, 0) = 0 dx - dy$
 $e^{0.04^3-0.01} \approx 1 - 0.01 = 0.99$
3. $\arcsin \frac{0.47}{1.04}$
 $x_0 = \frac{1}{2}, y_0 = 1, dx = -0.03, dy = 0.04$
 $f(x, y) = \arcsin \frac{x}{y}, \quad f(x_0, y_0) = \arcsin \frac{1}{2} = \frac{\pi}{6}$
 $f_x = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \frac{1}{y} = \frac{1}{y\sqrt{y^2-x^2}}, \quad f_y = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2-x^2}}$
 $df(x, y) = \frac{1}{y\sqrt{y^2-x^2}} dx - \frac{x}{y\sqrt{y^2-x^2}} dy, \quad df\left(\frac{1}{2}, 1\right) = \frac{2}{\sqrt{3}} dx - \frac{1}{\sqrt{3}} dy$
 $\arcsin \frac{0.47}{1.04} \approx \frac{\pi}{6} + \frac{2}{\sqrt{3}} \cdot (-0.03) - \frac{1}{\sqrt{3}} \cdot 0.04 = \frac{\pi}{6} - \frac{1}{10\sqrt{3}}$

Spočtete rovnici tečné roviny a normály ke grafu funkce f v bodě (x_0, y_0)

$$\tau : z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$n : \frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - f(x_0, y_0)}{-1}$$

1. $f(x, y) = x^3 + 3xy^2 - xy + x$, $(x_0, y_0) = (1, 0)$
 $f_x = 3x^2 + 3y^2 - y + 1$, $f_x(1, 0) = 4$, $f_y = 6xy - x$, $f_y(1, 0) = -1$
 $f(1, 0) = 1 + 0 - 0 + 1 = 2$
 $\tau : z = 2 + 4(x - 1) - (y - 0) \Rightarrow 4x - y - z - 2 = 0$
parametrické vyjádření normály: $n : [x, y, z] = [1, 0, 2] + t(4, -1, -1)$, $t \in R$
obecné vyjádření normály: $\frac{x-1}{4} = \frac{y}{-1} = \frac{z-2}{-1}$
2. $f(x, y) = \ln(x + 2y)$, $(x_0, y_0) = (2, 1)$
 $f_x = \frac{1}{x+2y}$, $f_x(2, 1) = \frac{1}{4}$, $f_y = \frac{2}{x+2y}$, $f_y(2, 1) = \frac{1}{2}$
 $f(2, 1) = \ln 4$
 $\tau : z = \ln 4 + \frac{1}{4}(x - 2) + \frac{1}{2}(y - 1) \Rightarrow x + 2y - 4z + 4(\ln 4 - 1) = 0$
 $n : \frac{x-2}{\frac{1}{4}} = \frac{y-1}{\frac{1}{2}} = \frac{z-\ln 4}{-1}$
 $n : 4(x - 2) = 2(y - 1) = \ln 4 - z$
 $n : [x, y, z] = [2, 1, \ln 4] + t(1, 2, -4)$, $t \in R$

Pomocí gradientu vypočtete derivaci funkce f v bodě a ve směru $\vec{u} = (u_1, u_2)$

$$\nabla(x, y) = (f_x(x, y), f_y(x, y)), \quad f_u = \nabla(x, y) \cdot \vec{u} = f_x(x, y)u_1 + f_y(x, y)u_2$$

1. $f(x, y) = \ln(x^2 + y^2)$, $a = (5, -2)$, $\vec{u} = (-5, 2)$
 $f_x = \frac{2x}{x^2+y^2}$, $f_y = \frac{2y}{x^2+y^2}$, $\nabla(x, y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$, $\nabla(5, -2) = \left(\frac{10}{29}, \frac{-4}{29}\right)$
 $f_u(5, -2) = \nabla(5, -2) \cdot \vec{u} = \left(\frac{10}{29}, \frac{-4}{29}\right) \cdot (-5, 2) = -\frac{50}{29} - \frac{8}{29} = -\frac{58}{29} = -2$
2. $f(x, y) = \arctg(xy)$, $a = (1, 1)$, $\vec{u} = (\sqrt{2}, \sqrt{2})$
 $f_x = \frac{y}{1+x^2y^2}$, $f_y = \frac{x}{1+x^2y^2}$, $\nabla(x, y) = \left(\frac{y}{1+x^2y^2}, \frac{x}{1+x^2y^2}\right)$, $\nabla(1, 1) = \left(\frac{1}{2}, \frac{1}{2}\right)$
 $f_u(a) = \nabla(1, 1) \cdot \vec{u} = \left(\frac{1}{2}, \frac{1}{2}\right) \cdot (\sqrt{2}, \sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$
3. $f(x, y, z) = xy^2 - z^3 + xyz$, $a = (-1, 1, 2)$, $\vec{u} = (1, 0, -1)$
 $f_x = y^2 + yz$, $f_y = 2xy + xz$, $f_z = -3z^2 + xy$
 $\nabla(x, y, z) = (y^2 + yz, 2xy + xz, -3z^2 + xy)$, $\nabla(-1, 1, 2) = (3, -4, -13)$
 $f_u(a) = (3, -4, -13) \cdot (1, 0, -1) = 3 + 13 = 16$

Nalezněte diferenciál druhého řádu funkce f

$$d^2f(x, y) = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

1. $f(x, y) = x^3y - xy^3$
 $f_x = 3x^2y - y^3$, $f_y = x^3 - 3xy^2$
 $f_{xx} = 6xy$, $f_{yy} = -6xy$, $f_{xy} = 3x^2 - 3y^2$
 $d^2f(x, y) = 6xy dx^2 + (6x^2 - 6y^2) dx dy - 6xy dy^2$
2. $f(x, y) = e^y \cos x$, $(x_0, y_0) = (0, 0)$
 $f_x = e^y(-\sin x)$, $f_y = e^y \cos x$
 $f_{xx} = -e^y \cos x$, $f_{yy} = e^y \cos x$, $f_{xy} = -e^y \sin x$
 $d^2f(x, y) = -e^y \cos x dx^2 - 2e^y \sin x dx dy + e^y \cos x dy^2$
 $d^2f(x_0, y_0) = -1 dx^2 - 0 dx dy + 1 dy^2$
3. $f(x, y, z) = xyz$
 $f_x = yz$, $f_y = xz$, $f_z = xy$
 $f_{xx} = 0$, $f_{xy} = z$, $f_{yy} = 0$, $f_{yz} = x$, $f_{zz} = 0$, $f_{xz} = y$
 $d^2f(x, y, z) = 0 dx^2 + 0 dy^2 + 0 dz^2 + 2(z dx dy + x dy dz + y dx dz)$

Nalezněte Taylorův polynom řádu k funkce f v bodě (x_0, y_0)

$$T_k(x, y) = f(x_0, y_0) + \frac{1}{1!}df(x_0, y_0) + \frac{1}{2!}d^2f(x_0, y_0) + \dots + \frac{1}{k!}d^k f(x_0, y_0)$$

$$d^k f(x, y) = \sum_{j=0}^k \binom{k}{j} \frac{\partial^k f}{\partial x^{k-j} \partial y^j}(x, y) dx^{k-j} dy^j, \quad dx = x - x_0, dy = y - y_0$$

1. $f(x, y) = \sqrt{x^2 + y^2}$, $(x_0, y_0) = (4, 3)$, $k = 2$

$$f(4, 3) = \sqrt{16 + 9} = 5$$

$$f_x = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_x(4, 3) = \frac{4}{5}$$

$$f_y = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}, \quad f_y(4, 3) = \frac{3}{5}$$

$$f_{xx} = \frac{\sqrt{x^2 + y^2} - x \frac{2x}{2\sqrt{x^2 + y^2}}}{x^2 + y^2} = \frac{y^2}{\sqrt{x^2 + y^2}^3}, \quad f_{xx}(4, 3) = \frac{9}{125}$$

$$f_{yy} = \frac{\sqrt{x^2 + y^2} - y \frac{2y}{2\sqrt{x^2 + y^2}}}{x^2 + y^2} = \frac{x^2}{\sqrt{x^2 + y^2}^3}, \quad f_{yy}(4, 3) = \frac{16}{125}$$

$$f_{xy} = -\frac{1}{2} \frac{x}{\sqrt{x^2 + y^2}^3} 2y = \frac{-xy}{\sqrt{x^2 + y^2}^3}, \quad f_{xy} = -\frac{12}{125}$$

$$T_2(x, y) = 5 + \frac{4}{5}(x - 4) + \frac{3}{5}(y - 3) + \frac{1}{2} \left(\frac{9}{125}(x - 4)^2 - 2 \frac{12}{125}(x - 4)(y - 3) + \frac{16}{125}(y - 3)^2 \right)$$

$$= 5 + \frac{4}{5}(x - 4) + \frac{3}{5}(y - 3) + \frac{9}{250}(x - 4)^2 - \frac{12}{125}(x - 4)(y - 3) + \frac{8}{125}(y - 3)^2$$

2. $f(x, y) = \sin(x^2 + y)$, $(x_0, y_0) = (0, \frac{\pi}{4})$, $k = 2$

$$f(0, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$f_x = \cos(x^2 + y)2x, \quad f_x(0, \frac{\pi}{4}) = 0$$

$$f_y = \cos(x^2 + y), \quad f_y(0, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$f_{yy} = -\sin(x^2 + y), \quad f_{yy}(0, \frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$$

$$f_{xy} = -\sin(x^2 + y)2x, \quad f_{xy}(0, \frac{\pi}{4}) = 0$$

$$f_{xx} = -\sin(x^2 + y)2x + \cos(x^2 + y)2,$$

$$f_{xx}(0, \frac{\pi}{4}) = \sqrt{2}$$

$$T_2(x, y) = \frac{\sqrt{2}}{2} + 0x + \frac{\sqrt{2}}{2} \left(y - \frac{\pi}{4} \right) + \frac{1}{2} \left(\sqrt{2}x^2 + 2 \cdot 0x \left(y - \frac{\pi}{4} \right) - \frac{\sqrt{2}}{2} \left(y - \frac{\pi}{4} \right)^2 \right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \left(y - \frac{\pi}{4} \right) + \frac{1}{2} \left(\sqrt{2}x^2 - \frac{\sqrt{2}}{2} \left(y - \frac{\pi}{4} \right)^2 \right)$$

3. $f(x, y) = \ln(1 + x) \ln(1 + y)$, $(x_0, y_0) = (0, 0)$, $k = 3$

$$f(0, 0) = 0$$

$$f_x = \frac{1}{1+x} \ln(1 + y), \quad f_x(0, 0) = 0$$

$$f_y = \ln(1 + x) \frac{1}{1+y}, \quad f_y(0, 0) = 0$$

$$f_{xx} = -\frac{1}{(1+x)^2} \ln(1 + y), \quad f_{xx}(0, 0) = 0$$

$$f_{xy} = \frac{1}{1+x} \frac{1}{1+y}, \quad f_{xy}(0, 0) = 1$$

$$f_{yy} = -\ln(1 + x) \frac{1}{(1+y)^2}$$

$$f_{xxx} = 2 \frac{1}{(1+x)^3} \ln(1 + y), \quad f_{xxx}(0, 0) = 0$$

$$f_{xxy} = -\frac{1}{(1+x)^2} \frac{1}{1+y}, \quad f_{xxy}(0, 0) = -1$$

$$f_{yyy} = 2 \ln(1 + x) \frac{1}{(1+y)^3}, \quad f_{yyy}(0, 0) = 0$$

$$f_{yyx} = -\frac{1}{1+x} \frac{1}{(1+y)^2}, \quad f_{yyx}(0, 0) = -1$$

$$T_3(x, y) = 0 + 0x + 0y + \frac{1}{2}(0x^2 + 2 \cdot 1xy + 0y^2) + \frac{1}{6}(0x^3 + 3(-1)x^2y + 3(-1)xy^2 + 0y^3)$$

$$= xy - \frac{1}{2}x^2y - \frac{1}{2}xy^2$$

1. Vypočtete diferenciál $df(x)$ pro funkci $f(x) = \arctan \frac{1}{x}$, $h \in \mathbb{R}$.

Řešení:

$$f'(x) = \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{1 + x^2},$$

$$df(x) = f'(x)h = -\frac{1}{1 + x^2}h.$$

2. Vypočtete přírůstek funkce $f(x_0 + h) - f(x_0)$ a diferenciál funkce $df(x_0)$, je-li $f(x) = x^2 - 3x + 4$, $x_0 = 1$, $h = -0,2$.

Řešení:

$$f(x_0 + h) - f(x_0) = f(0,8) - f(1) = (0,8^2 - 3 \cdot 0,8 + 4) - (1^2 - 3 \cdot 1 + 4) = 2,24 - 2 = 0,24.$$

$$df(x_0) = f'(x_0)h = (2x_0 - 3)h = (2 \cdot 1 - 3) \cdot (-0,2) = 0,2.$$

3. Užitím diferenciálu vypočtete přibližně

a) $2,03^5$,

b) $\cos 59^\circ$.

Řešení:

- a) Použijeme vzorec $f(x) \doteq f(x_0) + df(x_0) = f(x_0) + f'(x_0)(x - x_0)$. Máme

$$f(x) = x^5, x_0 = 2, x - x_0 = h = 0,03,$$

$$f(x_0) = f(2) = 2^5 = 32, f'(x) = 5x^4, f'(x_0) = f'(2) = 5 \cdot 2^4 = 80.$$

$$\text{Aplikujeme diferenciál: } 2,03^5 \doteq 32 + 80 \cdot 0,03 = 32 + 2,4 = 34,4.$$

Přesná hodnota je po zaokrouhlení 34,473.

- b) $f(x) = \cos x$, $x_0 = 60^\circ = \frac{\pi}{3}$, $x - x_0 = h = -1^\circ = -\frac{\pi}{180}$,

$$f(x_0) = \cos \frac{\pi}{3} = \frac{1}{2}, f'(x) = -\sin x, f'(x_0) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$\cos 59^\circ \doteq \frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right) \cdot \left(-\frac{\pi}{180}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot 0,0174 \doteq 0,5151.$$

Přesná hodnota je po zaokrouhlení 0,515.

4. Napište Taylorův polynom 5. stupně se středem v bodě $x_0 = 1$ pro funkci

$$f(x) = x^5 - 3x^4 + 2x^3 + 4x^2 - 6x + 5.$$

Řešení: Taylorův polynom stupně n funkce $f(x)$ se středem v bodě x_0 má tvar

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

$$f(x) = x^5 - 3x^4 + 2x^3 + 4x^2 - 6x + 5,$$

$$f'(x) = 5x^4 - 12x^3 + 6x^2 + 8x - 6,$$

$$f''(x) = 20x^3 - 36x^2 + 12x + 8,$$

$$f'''(x) = 60x^2 - 72x + 12,$$

$$f^{(4)}(x) = 120x - 72,$$

$$f^{(5)}(x) = 120.$$

Pro $x_0 = 1$ máme

$$f(1) = 3, f'(1) = 1, f''(1) = 4, f'''(1) = 0, f^{(4)}(1) = 48, f^{(5)}(1) = 120.$$

Dosazením do vzorce dostáváme

$$\begin{aligned} T_5(x) &= 3 + \frac{1}{1!}(x - 1) + \frac{4}{2!}(x - 1)^2 + \frac{0}{3!}(x - 1)^3 + \frac{48}{4!}(x - 1)^4 + \frac{120}{5!}(x - 1)^5 \\ &= 3 + (x - 1) + 2(x - 1)^2 + 2(x - 1)^4 + (x - 1)^5. \end{aligned}$$

5. Napište Taylorův polynom 4. stupně se středem v bodě $x_0 = -1$ pro funkci $f(x) = \frac{1}{x}$.

Řešení:

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4}, \quad f^{(4)}(x) = \frac{24}{x^5}.$$

Pro $x_0 = -1$:

$$f(-1) = -1, \quad f'(-1) = -1, \quad f''(-1) = -2, \quad f'''(-1) = -6, \quad f^{(4)}(-1) = -24.$$

$$\begin{aligned} T_4(x) &= -1 + \frac{-1}{1!}(x+1) + \frac{-2}{2!}(x+1)^2 + \frac{-6}{3!}(x+1)^3 + \frac{-24}{4!}(x+1)^4 \\ &= -1 - (x+1) - (x+1)^2 - (x+1)^3 - (x+1)^4. \end{aligned}$$

6. Napište Taylorův polynom 3. stupně se středem v bodě $x_0 = 0$ pro funkci $f(x) = e^{\cos x}$.

Řešení:

$$\begin{aligned} f(x) &= e^{\cos x}, \\ f'(x) &= (-\sin x) \cdot e^{\cos x}, \\ f''(x) &= (-\cos x) \cdot e^{\cos x} + (-\sin x) \cdot e^{\cos x} \cdot (-\sin x) = e^{\cos x} (\sin^2 x - \cos x), \\ f'''(x) &= e^{\cos x} \cdot (-\sin x) \cdot (\sin^2 x - \cos x) + e^{\cos x} \cdot (2 \sin x \cos x + \sin x) \\ &= e^{\cos x} (-\sin^3 x + 3 \sin x \cos x + \sin x) = e^{\cos x} \sin x (-\sin^2 x + 3 \cos x + 1). \end{aligned}$$

Pro $x_0 = 0$:

$$\begin{aligned} f(0) &= e^0 = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0. \\ T_3(x) &= 1 + \frac{0}{1!}(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 = 1 - \frac{x^2}{2}. \end{aligned}$$

7. Napište Taylorův polynom 3. stupně se středem v bodě $x_0 = 0$ pro funkci $f(x) = \arctan x$.

Řešení:

$$\begin{aligned} f(x) &= \arctan x, \\ f'(x) &= \frac{1}{1+x^2}, \\ f''(x) &= -(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2}, \\ f'''(x) &= \frac{-2 \cdot (1+x^2)^2 - (-2x) \cdot 2 \cdot (1+x^2) \cdot 2x}{(1+x^2)^4} = \frac{-2(1+x^2)(1+x^2-4x^2)}{(1+x^2)^4} \\ &= \frac{-2(1-3x^2)}{(1+x^2)^3}. \end{aligned}$$

Pro $x_0 = 0$:

$$\begin{aligned} f(0) &= 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -2. \\ T_3(x) &= 0 + \frac{1}{1!}(x-0) + \frac{0}{2!}(x-0)^2 + \frac{-2}{3!}(x-0)^3 = x - \frac{1}{3}x^3. \end{aligned}$$