

Vzorce na integrování

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
2. $\int \frac{1}{x} dx = \ln |x| + C$
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C$
5. $\int \sin x dx = -\cos x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + C$
8. $\int \frac{1}{\sin^2 x} dx = -\operatorname{cotg} x + C$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C = -\arccos x + C$
10. $\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + C = -\operatorname{arccotg} x + C$
11. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$
12. $\int \frac{1}{\sqrt{1+x^2}} dx = \ln(x + \sqrt{1+x^2}) + C$
13. $\int \frac{1}{\sqrt{x^2-1}} dx = \ln|x + \sqrt{x^2-1}| + C$

$$\int [af(x) + bg(x)] dx = a \int f(x) dx + b \int g(x) dx$$

Pozor !!!

Neplatí žádné univerzální pravidlo pro integrál součinu a podílu dvou funkcí!!!

Integruijete

1. $\int (3x^3 - x^2 - 3x + 5) dx = 3 \int x^3 dx - \int x^2 dx - 3 \int x dx + 5 \int dx = 3 \cdot \frac{x^4}{4} - \frac{x^3}{3} - 3 \cdot \frac{x^2}{2} + 5 \cdot x + C$
2. $\int (x+1)(x-1)(x^2-1) dx = \int (x^2-1)(x^2-1) dx = \int (x^4 - 2x^2 + 1) dx = \frac{x^5}{5} - 2\frac{x^3}{3} + x + C$
3. $\int \frac{x^5+4x^4-x^2+2}{x^2} dx = \int (x^3 + 4x^2 - 1 + \frac{2}{x^2}) dx = \frac{x^4}{4} + 4\frac{x^3}{3} - x - 2\frac{1}{x} + C$
4. $\int \frac{(x-1)^2}{x^3} dx = \int \frac{x^2-2x+1}{x^3} dx = \int (\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}) dx = \ln|x| + \frac{2}{x} - \frac{1}{2x^2} + C$
5. $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}} dx = \int (\frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}}} - \frac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}}) dx = \int (x^{\frac{1}{6}} - x^{-\frac{1}{4}}) dx = \frac{x^{\frac{7}{6}}}{\frac{7}{6}} - \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + C = \frac{6\sqrt[6]{x^7}}{7} - \frac{4\sqrt[4]{x^3}}{3} + C = \frac{6x\sqrt[6]{x}}{7} - \frac{4\sqrt[4]{x^3}}{3} + C$
6. $\int \frac{1}{\sqrt{3-3x^2}} dx = \int \frac{1}{\sqrt{3(1-x^2)}} dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1-x^2}} dx = \frac{1}{\sqrt{3}} \arcsin x + C = \frac{\sqrt{3}}{3} \arcsin x + C$
7. $\int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x} dx = \int (3 - 2(\frac{3}{2})^x) dx = 3x - 2 \cdot (\frac{3}{2})^x \frac{1}{\ln \frac{3}{2}} + C$
8. $\int \frac{1+\cos^2 x}{1+\cos 2x} dx = \int \frac{1+\cos^2 x}{1+\cos^2 x - \sin^2 x} dx = \int \frac{1+\cos^2 x}{2\cos^2 x} dx = \int \frac{1}{2\cos^2 x} dx + \int \frac{\cos^2 x}{2\cos^2 x} dx = \frac{1}{2} \operatorname{tg} x + \frac{1}{2} x + C$
9. $\int 2 \sin^2 \frac{x}{2} dx = \int (\sin^2 \frac{x}{2} + \sin^2 \frac{x}{2}) dx = \int (\sin^2 \frac{x}{2} + 1 - \cos^2 \frac{x}{2}) dx = \int (1 - (\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2})) dx = \int (1 - \cos x) dx = x - \sin x + C$

Integreze metodou per partes

1.

$$\int x \sin x dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \sin x & v = -\cos x \end{array} \right| = x \cdot (-\cos x) - \int 1 \cdot (-\cos x) dx = -x \cos x + \sin x + C$$

2.

$$\begin{aligned} \int x^2 e^x dx &= \left| \begin{array}{ll} u = x^2 & u' = 2x \\ v' = e^x & v = e^x \end{array} \right| = x^2 e^x - \int 2x e^x dx = x^2 e^x - 2 \int x e^x dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = e^x & v = e^x \end{array} \right| \\ &= x^2 e^x - 2[x e^x - \int 1 e^x dx] = x^2 e^x - 2x e^x + 2e^x + C = e^x(x^2 - 2x + 2) + C \end{aligned}$$

3.

$$\int \ln x dx = \int 1 \cdot \ln x dx = \left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array} \right| = x \ln x - \int \frac{1}{x} \cdot x dx = x \ln x - \int dx = x \ln x - x + C$$

4.

$$\begin{aligned} \int x \operatorname{arctg} x dx &= \left| \begin{array}{ll} u = \operatorname{arctg} x & u' = \frac{1}{1+x^2} \\ v' = x & v = \frac{x^2}{2} \end{array} \right| = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx \\ &= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arctg} x + C \end{aligned}$$

5.

$$\begin{aligned}\int e^x \cos x \, dx &= \left| \begin{array}{cc} u = e^x & u' = e^x \\ v' = \cos x & v = \sin x \end{array} \right| = e^x \sin x - \int e^x \sin x \, dx = \left| \begin{array}{cc} u = e^x & u' = e^x \\ v' = \sin x & v = -\cos x \end{array} \right| \\ &= e^x \sin x - [e^x(-\cos x) - \int e^x(-\cos x) \, dx] = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \\ 2 \int e^x \cos x \, dx &= e^x \sin x + e^x \cos x \\ \int e^x \cos x \, dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C\end{aligned}$$

6.

$$\begin{aligned}\int \sin^2 x \, dx &= \int \sin x \cdot \sin x \, dx = \left| \begin{array}{cc} u = \sin x & u' = \cos x \\ v' = \sin x & v = -\cos x \end{array} \right| \\ &= -\sin x \cos x + \int \cos^2 x \, dx = -\sin x \cos x + \int (1 - \sin^2 x) \, dx = -\sin x \cos x + \int dx - \int \sin^2 x \, dx\end{aligned}$$

Odtud

$$\begin{aligned}2 \int \sin^2 x \, dx &= -\sin x \cos x + x \\ \int \sin^2 x \, dx &= \frac{1}{2}(x - \sin x \cos x) + C\end{aligned}$$

Substituční metoda

1.

$$\int \frac{1}{1-x} \, dx = \left| \begin{array}{cc} 1-x & = \quad t \\ -dx & = \quad dt \end{array} \right| = \int \frac{1}{t} \frac{dt}{-1} = -\int \frac{1}{t} \, dt = -\ln |t| + C = -\ln |1-x| + C$$

2.

$$\int \frac{e^x}{2+e^x} \, dx = \left| \begin{array}{cc} 2+e^x & = \quad t \\ e^x dx & = \quad dt \end{array} \right| = \int \frac{e^x}{t} \frac{dt}{e^x} = \int \frac{1}{t} \, dt = \ln |t| + C = \ln |2+e^x| + C = \ln (2+e^x) + C$$

3.

$$\int \cotg x \, dx = \int \frac{\cos x}{\sin x} \, dx = \left| \begin{array}{cc} \sin x & = \quad t \\ \cos x dx & = \quad dt \end{array} \right| = \int \frac{1}{t} \, dt = \ln |t| + C = \ln |\sin x| + C$$

4.

$$\int \frac{2x}{\sqrt{x^2+1}} \, dx = \left| \begin{array}{cc} x^2+1 & = \quad t \\ 2x dx & = \quad dt \end{array} \right| = \int \frac{1}{\sqrt{t}} \, dt = \int t^{-\frac{1}{2}} \, dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{t} + C = 2\sqrt{x^2+1} + C$$

5.

$$\begin{aligned}\int \sqrt{-2x+8} \, dx &= \left| \begin{array}{cc} -2x+8 & = \quad t \\ -2dx & = \quad dt \end{array} \right| = \int \sqrt{t} \frac{dt}{-2} = -\frac{1}{2} \int t^{\frac{1}{2}} \, dt = -\frac{1}{2} \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= -\frac{1}{3} \sqrt{t^3} + C = -\frac{1}{3} \sqrt{(-2x+8)^3} + C\end{aligned}$$

$$6. \quad \int \frac{\sqrt{\ln x}}{x} dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right| = \int \sqrt{t} dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3} \sqrt{\ln^3 x} + C$$

$$7. \quad \int \frac{\operatorname{arctg}^2 x}{1+x^2} dx = \left| \begin{array}{l} \operatorname{arctg} x = t \\ \frac{1}{1+x^2} dx = dt \end{array} \right| = \int t^2 dt = \frac{t^3}{3} + C = \frac{\operatorname{arctg}^3 x}{3} + C$$

$$8. \quad \int \frac{1}{2x-1} dx = \left| \begin{array}{l} 2x-1 = t \\ 2dx = dt \end{array} \right| = \int \frac{1}{t} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{t} dt = \frac{1}{2} \ln |t| + C = \frac{1}{2} \ln |2x-1| + C$$

$$9. \quad \int e^{-x^3} x^2 dx = \left| \begin{array}{l} -x^3 = t \\ -3x^2 dx = dt \end{array} \right| = \int e^t x^2 \frac{dt}{-3x^2} = -\frac{1}{3} \int e^t dt = -\frac{1}{3} e^t + C = -\frac{1}{3} e^{-x^3} + C$$

$$10. \quad \int \cos^3 x \sin x dx = \left| \begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array} \right| = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

$$11. \quad \int \frac{1}{1+9x^2} dx = \int \frac{1}{1+(3x)^2} dx = \left| \begin{array}{l} 3x = t \\ 3dx = dt \end{array} \right| = \int \frac{1}{1+t^2} \frac{dt}{3} = \frac{1}{3} \operatorname{arctg} t + C = \frac{1}{3} \operatorname{arctg} 3x + C$$

$$12. \quad \int \frac{1}{\sqrt{4-9x^2}} dx = \int \frac{1}{\sqrt{4(1-\frac{9x^2}{4})}} dx = \int \frac{1}{2\sqrt{1-(\frac{3x}{2})^2}} dx = \left| \begin{array}{l} \frac{3x}{2} = t \\ \frac{3}{2} dx = dt \end{array} \right| = \frac{1}{2} \int \frac{1}{\sqrt{1-t^2}} \frac{2dt}{3} \\ = \frac{1}{3} \int \frac{1}{\sqrt{1-t^2}} dt = \frac{1}{3} \arcsin t + C = \frac{1}{3} \arcsin \frac{3x}{2} + C$$

$$13. \quad \int \frac{1}{a^2+x^2} dx = \frac{1}{a^2} \int \frac{1}{1+\frac{x^2}{a^2}} dx = \left| \begin{array}{l} \frac{x}{a} = t \\ \frac{1}{a} dx = dt \end{array} \right| = \frac{1}{a^2} \int \frac{1}{1+t^2} \cdot a dt = \frac{1}{a} \operatorname{arctg} t + C = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$$

Převeďte na součet polynomu a ryze racionální funkce a integrujte

$$1. \quad \int \frac{x}{2x+1} dx = |\text{dělení}| = \int \left(\frac{1}{2} - \frac{1}{2} \frac{1}{2x+1} \right) dx = \frac{1}{2} x - \frac{1}{2} \int \frac{1}{2x+1} dx = \left| \begin{array}{l} 2x+1 = t \\ 2dx = dt \end{array} \right| \\ = \frac{1}{2} x - \frac{1}{2} \int \frac{1}{t} \frac{dt}{2} = \frac{1}{2} x - \frac{1}{4} \int \frac{1}{t} dt = \frac{1}{2} x - \frac{1}{4} \ln |t| + C = \frac{1}{2} x - \frac{1}{4} \ln |2x+1| + C$$

$$2. \quad \int \frac{x^2-1}{x^2+1} dx = \int \left(1 - \frac{2}{x^2+1} \right) dx = x - 2 \operatorname{arctg} x + C$$

3.

$$\begin{aligned}\int \frac{x^4}{1-x} dx &= \int \left(-x^3 - x^2 - x - 1 + \frac{1}{1-x} \right) dx = \left| \begin{array}{l} 1-x = t \\ -dx = dt \end{array} \right| = -\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \int \frac{1}{t} dt \\ &= -\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} - x - \ln|1-x| + C\end{aligned}$$

Rozložte ryze lomenou funkci na součet parciálních zlomků

1. $R(x) = \frac{2x+1}{x^3-x}$

Rozložíme jmenovatel na součin kořenových faktorů (v reálném oboru).

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$$

Protože máme tři různé jednonásobné reálné kořeny, rozklad dopadne takto:

$$\frac{2x+1}{x^3-x} = \frac{2x+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Vynásobíme rovnost společným jmenovatelem z levé strany.

$$2x+1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Máme dvě možnosti

a) do rovnice postupně dosadíme reálné kořeny původního jmenovatele:

$$\begin{array}{lll} x=0: & 0+1 = A(-1)(1) + B \cdot 0(1) + C \cdot 0(-1) & \Rightarrow A = -1 \\ x=1: & 2+1 = B \cdot 1(1+1) & \Rightarrow B = \frac{3}{2} \\ x=-1: & -2+1 = C(-1)(-2) & \Rightarrow C = -\frac{1}{2} \end{array}$$

b) rovnici ještě upravíme a porovnáme koeficienty u jednotlivých mocnin x na pravé a na levé straně:

$$\begin{aligned} 2x+1 &= Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx \\ 2x+1 &= x^2(A+B+C) + x(B-C) - A \\ x^2: & 0 = A+B+C \\ x^1: & 2 = B-C \\ x^0: & 1 = -A \end{aligned}$$

Řešením soustavy tří rovnic o třech neznámých dostaneme $A = -1$, $B = \frac{3}{2}$, $C = -\frac{1}{2}$.

Celkem máme:

$$R(x) = \frac{-1}{x} + \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}.$$

2. $R(x) = \frac{x^2-x-1}{x^3-2x^2+x}$

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

Máme jeden jednonásobný kořen $x = 0$ a dvojnásobný kořen $x = 1$. Rozklad bude vypadat takto:

$$\frac{x^2-x-1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2 - x - 1 = A(x-1)^2 + Bx(x-1) + Cx$$

$$x^2 - x - 1 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$$

Koeficienty najdeme kombinací obou metod. Nejprve dosadíme dva reálné kořeny a potom poslední koeficient získáme metodou porovnávání koeficientů.

$$x = 0 : \quad -1 = A \Rightarrow A = -1$$

$$x = 1 : \quad -1 = C \Rightarrow C = -1$$

Porovnáme koeficient u x^1 :

$$-1 = -2A - B + C \Rightarrow B = 2$$

A tedy

$$R(x) = -\frac{1}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2}.$$

$$3. \quad R(x) = \frac{2x^3+x+2}{x^4+x^3+x^2}$$

$$x^4 + x^3 + x^2 = x^2(x^2 + x + 1)$$

Máme dvojnásobný reálný kořen $x = 0$ a kvadratický dvojčlen se záporným diskriminantem.

$$\frac{2x^3 + x + 2}{x^2(x^2 + x + 1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx + D}{x^2 + x + 1}$$

$$2x^3 + x + 2 = A(x^2 + x + 1) + Bx(x^2 + x + 1) + (Cx + D)x^2$$

$$2x^3 + x + 2 = Ax^2 + Ax + A + Bx^3 + Bx^2 + Bx + Cx^3 + Dx^2$$

$$2x^3 + x + 2 = x^3(B + C) + x^2(A + B + D) + x(A + B) + A$$

Porovnáním koeficientů:

$$\begin{array}{rclcl} & B & + & C & = & 2, \\ A & + & B & & + & D & = & 0, \\ A & + & B & & & & = & 1, \\ A & & & & & & = & 2. \end{array}$$

Odtud $A = 2, B = -1, C = 3, D = -1$, tj.

$$R(x) = \frac{2}{x^2} - \frac{1}{x} + \frac{3x-1}{x^2+x+1}.$$

Integrovaní parciálních zlomků

I. typ: $\int \frac{A}{x-\alpha} dx$

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$$\int \frac{3}{x-5} dx = \left| \begin{array}{lcl} x-5 & = & t \\ dx & = & dt \end{array} \right| = 3 \int \frac{1}{t} dt = 3 \ln |x-5| + C$$

II. typ: $\int \frac{A}{(x-\alpha)^k} dx, \quad k \neq -1$

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$$\int \frac{\sqrt{2}}{(x+2)^5} dx = \left| \begin{array}{lcl} x+2 & = & t \\ dx & = & dt \end{array} \right| = \sqrt{2} \int \frac{1}{t^5} dt = \sqrt{2} \int t^{-5} dt = \frac{\sqrt{2}}{-4} \frac{1}{t^4} + C = -\frac{\sqrt{2}}{4} \frac{1}{(x+2)^4} + C$$

III. typ: $\int \frac{Bx+C}{x^2+px+q} dx$, $p^2 - 4q < 0$

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$$\int \frac{2x-3}{x^2-3x+5} dx = \left| \begin{array}{lcl} x^2-3x+5 & = & t \\ (2x-3)dx & = & dt \end{array} \right| = \int \frac{1}{t} dt = \ln|t| + C = \ln|x^2-3x+5| + C$$

•

$$\begin{aligned} \int \frac{1}{x^2-2x+3} dx &= \int \frac{1}{x^2-2x+1-1+3} dx = \int \frac{1}{(x-1)^2+2} dx = \int \frac{1}{2\left[\frac{(x-1)^2}{2}+1\right]} dx \\ &= \frac{1}{2} \int \frac{1}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1} dx = \left| \begin{array}{lcl} \frac{x-1}{\sqrt{2}} & = & t \\ \frac{1}{\sqrt{2}}dx & = & dt \end{array} \right| = \frac{1}{2} \int \frac{1}{t^2+1} \sqrt{2} dt = \frac{\sqrt{2}}{2} \int \frac{1}{t^2+1} dt \\ &= \frac{\sqrt{2}}{2} \operatorname{arctg} t + C = \frac{\sqrt{2}}{2} \operatorname{arctg} \left(\frac{x-1}{\sqrt{2}} \right) + C \end{aligned}$$

•

$$\begin{aligned} \int \frac{3x-1}{x^2+x+1} dx &= 3 \int \frac{x-\frac{1}{3}}{x^2+x+1} dx = \frac{3}{2} \int \frac{2x-\frac{2}{3}}{x^2+x+1} dx = \frac{3}{2} \int \frac{2x+1-1-\frac{2}{3}}{x^2+x+1} dx \\ &= \frac{3}{2} \int \frac{2x+1}{x^2+x+1} dx - \frac{3}{2} \int \frac{\frac{5}{3}}{x^2+x+1} dx = \frac{3}{2} \ln|x^2+x+1| - \frac{5}{2} \int \frac{1}{x^2+x+1} dx \\ &\quad \int \frac{1}{x^2+x+1} dx = \int \frac{1}{x^2+x+\frac{1}{4}-\frac{1}{4}+1} dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} dx \end{aligned}$$

Máme dvě možnosti

$$\begin{aligned} a) &= \frac{4}{3} \int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)^2+1} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} dx = \left| \begin{array}{lcl} \frac{2x+1}{\sqrt{3}} & = & t \\ \frac{2}{\sqrt{3}}dx & = & dt \end{array} \right| = \frac{4}{3} \int \frac{1}{t^2+1} \frac{\sqrt{3}}{2} dt \\ &= \frac{4}{3} \frac{\sqrt{3}}{2} \operatorname{arctg} t + C = \frac{2\sqrt{3}}{3} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C \end{aligned}$$

$$b) = \left| \begin{array}{lcl} x+\frac{1}{2} & = & t \\ dx & = & dt \end{array} \right| = \frac{3}{4} \int \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} dt = \left| \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \right| = \frac{2}{\sqrt{3}} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C$$

Celkem máme

$$\begin{aligned} \int \frac{3x-1}{x^2+x+1} dx &= \frac{3}{2} \ln|x^2+x+1| - \frac{5}{2} \frac{2\sqrt{3}}{3} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C \\ &= \frac{3}{2} \ln|x^2+x+1| - \frac{5\sqrt{3}}{3} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C \end{aligned}$$

Ryze lomenou funkci převed'te na součet parciálních zlomků a integrujte

1.

$$\int \frac{2x+1}{x^3-x} dx = \int \left(\frac{-1}{x} + \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1} \right) dx = -\ln|x| + \frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$$

2.

$$\int \frac{x^2-x-1}{x^3-2x^2+x} dx = \int \left(-\frac{1}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2} \right) dx = -\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + C$$

3.

$$\begin{aligned} \int \frac{2x^3+x+2}{x^4+x^3+x^2} dx &= \int \left(\frac{2}{x^2} - \frac{1}{x} + \frac{3x-1}{x^2+x+1} \right) dx \\ &= -2\frac{1}{x} - \ln|x| + \frac{3}{2} \ln|x^2+x+1| - \frac{5\sqrt{3}}{3} \operatorname{arctg} \left(\frac{2x+1}{\sqrt{3}} \right) + C \end{aligned}$$

(viz dříve)

4.

$$\int \frac{x^2+2x}{x^4+2x^2-3} dx$$

Rozložíme na parciální zlomky:

$$\frac{x^2+2x}{x^4+2x^2-3} = |x^2=y| = \frac{x^2+2x}{y^2+2y-3} = \frac{x^2+2x}{(y-1)(y+3)} = \frac{x^2+2x}{(x^2-1)(x^2+3)} = \frac{x^2+2x}{(x-1)(x+1)(x^2+3)}$$

$$\frac{x^2+2x}{(x-1)(x+1)(x^2+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+3}$$

$$x^2+2x = A(x+1)(x^2+3) + B(x-1)(x^2+3) + (Cx+D)(x-1)(x+1)$$

$$\begin{array}{lll} x=1: & 1+2=8A & \Rightarrow A=\frac{3}{8} \\ x=-1: & 1-2=-8B & \Rightarrow B=\frac{1}{8} \\ x^0: & 0=3A-3B-D & \Rightarrow D=\frac{3}{4} \\ x^3: & 0=A+B+C & \Rightarrow C=-\frac{1}{2} \end{array}$$

$$\frac{x^2+2x}{(x-1)(x+1)(x^2+3)} = \frac{\frac{3}{8}}{x-1} + \frac{\frac{1}{8}}{x+1} + \frac{-\frac{1}{2}x + \frac{3}{4}}{x^2+3}$$

$$\begin{aligned} \int \frac{x^2+2x}{x^4+2x^2-3} dx &= \int \left(\frac{3}{8} \frac{1}{x-1} + \frac{1}{8} \frac{1}{x+1} - \frac{1}{4} \frac{2x-3}{x^2+3} \right) dx = \frac{3}{8} \ln|x-1| + \frac{1}{8} \ln|x+1| - \frac{1}{4} \int \frac{2x-3}{x^2+3} dx \\ &= \frac{3}{8} \ln|x-1| + \frac{1}{8} \ln|x+1| - \frac{1}{4} \int \left(\frac{2x}{x^2+3} - \frac{3}{x^2+3} \right) dx \\ &= \frac{3}{8} \ln|x-1| + \frac{1}{8} \ln|x+1| - \frac{1}{4} \ln(x^2+3) + \frac{3}{4\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + C \end{aligned}$$

5.

$$\int \frac{x^4}{x^3-1} dx = \left| x^4 : (x^3-1) = x + \frac{x}{x^3-1} \right| = \int \left(x + \frac{x}{x^3-1} \right) dx$$

Ryze lomenou funkci $\frac{x}{x^3-1}$ rozložíme na součet parciálních zlomků

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$x^2: \quad 0 = A+B \quad \Rightarrow \quad A = -B$$

$$x^0: \quad 0 = A-C \quad \Rightarrow \quad A = C$$

$$x^1: \quad 1 = A-B+C \quad \Rightarrow \quad 1 = A+A+A$$

Řešením soustavy jsou koeficienty $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{1}{3}$. Pokračujeme v integrování

$$\int \frac{x^4}{x^3-1} dx = \int \left(x + \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} \right) dx$$

integrál $\int \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} dx$ spočítáme zvlášť:

$$\begin{aligned} \int \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} dx &= -\frac{1}{3} \int \frac{x-1}{x^2+x+1} dx = -\frac{1}{6} \int \frac{2x-2}{x^2+x+1} dx = -\frac{1}{6} \int \frac{2x+1-1-2}{x^2+x+1} dx \\ &= -\frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx - \frac{1}{6} \int \frac{-3}{x^2+x+1} dx = -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \int \frac{1}{x^2+x+1} dx \\ &= -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \int \frac{1}{(x^2+x+\frac{1}{4}) + \frac{3}{4}} dx = -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx \\ &= \left| \begin{array}{l} x + \frac{1}{2} = t \\ dx = dt \end{array} \right| = -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \int \frac{1}{t^2 + \frac{3}{4}} dt = -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{2} \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t}{\sqrt{3}} + C \\ &= -\frac{1}{6} \ln|x^2+x+1| + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C \end{aligned}$$

Celkem

$$\begin{aligned} \int \frac{x^4}{x^3-1} dx &= \int \left(x + \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} \right) dx \\ &= \frac{x^2}{2} + \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C \end{aligned}$$

Některé integrály typu $\int \sin^n x \cos^m x dx$

1.

$$\begin{aligned} \int \sin^3 x \cos^4 x dx &= \int \sin^2 x \cos^4 x \sin x dx = \int (1-\cos^2 x) \cos^4 x \sin x dx = \left| \begin{array}{l} \cos x = t \\ -\sin x dx = dt \end{array} \right| \\ &= -\int (1-t^2)t^4 dt = \int (t^6 - t^4) dt = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C \end{aligned}$$

$$\begin{aligned}
2. \quad \int \sin^5 x \, dx &= \int (1 - \cos^2 x)^2 \sin x \, dx = \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \end{array} \right| = - \int (1 - t^2)^2 \, dt \\
&= - \int (1 - 2t^2 + t^4) \, dt = -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

$$\begin{aligned}
3. \quad \int \sin^5 x \cos^3 x \, dx &= \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^5(1 - t^2) \, dt = \int (t^5 - t^7) \, dt \\
&= \frac{1}{6}t^6 - \frac{1}{8}t^8 + C = \frac{1}{6}\sin^6 x - \frac{1}{8}\sin^8 x + C
\end{aligned}$$

Lze použít i substituci $t = \cos x$.

$$\begin{aligned}
4. \quad \int \frac{\cos^3 x}{\sin^5 x} \, dx &= \int \frac{\cos^2 x}{\sin^5 x} \cos x \, dx = \int \frac{1 - \sin^2 x}{\sin^5 x} \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| \\
&= \int \frac{1 - t^2}{t^5} \, dt = \int \left(\frac{1}{t^5} - \frac{1}{t^3} \right) \, dt = -\frac{1}{4t^4} + \frac{1}{2t^2} + C = -\frac{1}{4\sin^4 x} + \frac{1}{2\sin^2 x} + C
\end{aligned}$$

$$5. \quad \int \sin^2 x \, dx = \left| \sin^2 x = \frac{1}{2}(1 - \cos 2x) \right| = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$6. \quad \int \cos^2 x \, dx = \left| \cos^2 x = \frac{1}{2}(1 + \cos 2x) \right| = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$\begin{aligned}
7. \quad \int \sin^4 x \cos^2 x \, dx &= \int \left[\frac{1}{2}(1 - \cos 2x) \right]^2 \cdot \frac{1}{2}(1 + \cos 2x) \, dx = \left| \begin{array}{l} 2x = t \\ 2 \, dx = dt \end{array} \right| \\
&= \frac{1}{16} \int (1 - \cos t - \cos^2 t + \cos^3 t) \, dt = \frac{1}{16} \int (\sin^2 t - \cos t + \cos^3 t) \, dt \\
&= \frac{1}{16} \left(\int \frac{1}{2}(1 - \cos 2t) \, dt - \int \cos t \, dt + \int (1 - \sin^2 t) \cos t \, dt \right) \\
&= \frac{1}{16} \left(\frac{1}{2}t - \frac{1}{4}\sin 2t - \sin t + \sin t - \frac{1}{3}\sin^3 t \right) + C = -\frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + \frac{1}{16}x + C
\end{aligned}$$

$$8. \quad \int \frac{1}{\sin x \cos x} \, dx = \int \frac{1}{\operatorname{tg} x \cos^2 x} \, dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ \frac{1}{\cos^2 x} \, dx = dt \end{array} \right| = \int \frac{1}{t} \, dt = \ln |\operatorname{tg} x| + C$$

$$\begin{aligned}
9. \quad \int \frac{1}{\sin x} \, dx &= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \, dx = \frac{1}{2} \int \frac{1}{\operatorname{tg} \frac{x}{2} \cos^2 \frac{x}{2}} \, dx = \left| \begin{array}{l} \operatorname{tg} \frac{x}{2} = t \\ \frac{1}{\cos^2 \frac{x}{2}} \, dx = dt \end{array} \right| \\
&= \int \frac{1}{t} \, dt = \ln |\operatorname{tg} \frac{x}{2}| + C
\end{aligned}$$

10.

$$\begin{aligned}\int \frac{1}{\cos x} dx &= \left| \cos x = \sin \left(x + \frac{\pi}{2}\right) \right| = \int \frac{1}{\sin \left(x + \frac{\pi}{2}\right)} dx = \left| \begin{array}{l} x + \frac{\pi}{2} = t \\ dx = dt \end{array} \right| = \int \frac{1}{\sin t} dt \\ &= \ln \left| \operatorname{tg} \left(\frac{1}{2} \left(x + \frac{\pi}{2}\right) \right) \right| + C\end{aligned}$$

Obecná goniometrická substituce $\operatorname{tg} \frac{x}{2} = t$

$$\operatorname{tg} \frac{x}{2} = t, \quad dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

1.

$$\begin{aligned}\int \frac{1 - \sin x}{1 + \cos x} dx &= \int \frac{1 - \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{1+t^2-2t}{1+t^2+1-t^2} \frac{2}{1+t^2} dt = \int \frac{1+t^2-2t}{1+t^2} dt \\ &= \int \left(1 - \frac{2t}{1+t^2} \right) dt = t - \ln(1+t^2) + C = \operatorname{tg} \frac{x}{2} - \ln \left(1 + \operatorname{tg}^2 \frac{x}{2} \right) + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{dx}{3 \sin x - 4 \cos x} &= \int \frac{1}{3 \frac{2t}{1+t^2} + 4 \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{2}{6t+4-4t^2} dt = \int \frac{1}{-2t^2+3t+2} dt \\ &= -\frac{1}{2} \int \frac{1}{t^2 - \frac{3}{2}t - 1} dt = -\frac{1}{2} \int \frac{1}{(t-2)(t+\frac{1}{2})} dt = -\frac{1}{2} \int \frac{\frac{2}{5}}{t-2} - \frac{\frac{2}{5}}{t+\frac{1}{2}} dt \\ &= -\frac{1}{5} \ln |t-2| + \frac{1}{5} \ln \left| t + \frac{1}{2} \right| + C = -\frac{1}{5} \ln \left| \operatorname{tg} \frac{x}{2} - 2 \right| + \frac{1}{5} \ln \left| \operatorname{tg} \frac{x}{2} + \frac{1}{2} \right| + C\end{aligned}$$